On Hardy inequalities with a remainder term

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Abstract

In this paper we study some improvements of the classical Hardy inequality. We add to the right hand side of the inequality a term which depends on some Lorentz norms of u or of its gradient and we find the best values of the constants for remaining terms. In both cases we show that the problem of finding the optimal value of the constant can be reduced to a spherically symmetric situation. This result is new when the right hand side is a Lorentz norm of the gradient.

1 Introduction

The classical Hardy inequality asserts that (see [22] and [23])

$$\int_\Omega |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx, \quad \forall u \in H_0^1(\Omega), \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^N$ containing the origin, $N > 2$. The constant in (1) is the best possible; however it is not attained. This fact allows to add to the right hand side of (1) a suitable remaining term involving some norm of u or of the gradient of u. The first result in this direction was obtained by Brezis and Vazquez in [10]; if $N > 2$ they show the so called *Hardy-Poincarè inequality*

$$\int_\Omega |\nabla u|^2 \, dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx \geq \frac{\Lambda_2}{R_\Omega^2} \|u\|_2^2, \ \forall u \in H_0^1(\Omega) \tag{2}$$

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where Λ_2 denotes the first eigenvalue of the Laplace operator in the unit disk of $\mathbb{R}^2$ and R_Ω is the radius of the ball $\Omega^\# \subseteq \mathbb{R}^N$ centered at the origin having the same measure as Ω. The constant in (2) is *optimal* even if it is again not achieved.

The first aim of this paper is to find the best value of the constant C in inequalities of the type (2) that involve a Lorentz norm of u as a remainder term. In particular we focus our attention to the following two inequalities:

$$\int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2} dx \geq C(|\Omega|)\,\|u\|^2_{\frac{2N}{N-1},2}\,,\ \ \forall u \in H^1_0(\Omega) \qquad (3)$$

$$\int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}\, dx \geq C(p,|\Omega|)\,\|u\|^2_{p,1}\,,\ \ \forall u \in H^1_0(\Omega) \qquad (4)$$

where $N > 2$ and $1 \leq p < 2^*$, $2^* = 2N/(N-2)$. We recall that u belongs to the Lorentz space $L(r,s)(\Omega)$ with $0 < r, s < +\infty$, if the quantity defined by

$$\|u\|_{r,s} = \omega_N^{1/r-1/s}\left(\int_{\Omega^\#}\left[u^\#(x)\,|x|^{N/r}\right]^s \frac{dx}{|x|^N}\right)^{1/s}$$

is finite, where $u^\#$ is the *spherical decreasing rearrangement* of u, which is a spherically symmetric function defined on $\Omega^\#$, decreasing along the radius, having the same distribution function as u. Inequalities (3) and (4) are not new (see [11], [19]). At least for (3), the best value of the constant might be also obtained using the results of [19]. The technique used to get (2) (or any of the inequalities we quoted before) follows a usual procedure. First we observe that we can restrict our attention to spherically symmetric functions defined on $\Omega^\#$. This can be done replacing u by $u^\#$. Indeed, this operation decreases the left hand side of inequality (2), since it it decreases the $H^1_0(\Omega)$-norm of u by the classical Pólya-Szegö principle (see [25]) and, by Hardy-Littlewood inequality (see [23], [7]), it increases the weighted L^2-norm of u with weight $|x|^{-2}$. Moreover, this operation does not change the norm of u in Lebesgue spaces or, more in general in Lorentz spaces . Once reduced the problem to spherically symmetric case, the best value of the constant in (2) is obtained by using what Brezis and Vazquez (see [10]) call the *magical transformation*

$$v(r) = u(r) r^{\frac{N-2}{2}} \qquad r = |x|. \qquad (5)$$

This transformation produces a dimension reduction of the problem from N to 2 dimensions which also explain the presence of the constant Λ_2 in the inequality (2).

In view of inequality (2), one can also wonder if it may be possible to replace the L^2-norm of u by a norm of the gradient ∇u of u. This question was showed to have a positive answer too. Indeed, Vazquez and Zuazua (see [28]) proved the following *Improved Hardy-Poincarè inequality*

$$\int_\Omega |\nabla u|^2\, dx - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2} dx \geq C(q,|\Omega|)\,\||\nabla u|\|^2_q\,, \quad \forall u \in H^1_0(\Omega)\,, \qquad (6)$$

where Ω is a bounded open set of $\mathbb{R}^N$ containing the origin, $N > 2$, $1 \leq q < 2$ and $C(q, |\Omega|)$ is a constant depending only on q and Ω. However the question about finding the best value of the constant $C(q, |\Omega|)$ seems to be still open and it is not even known if it is possible, in order to find this best value, to restrict the attention to spherically symmetric functions defined on a ball. As regards this last question we prove that as for inequality (6) it is possible to reduce the problem to a spherically symmetric case using a suitable symmetrization procedure. In this situation, unlike the cases explained before, it is not useful to replace the function u by its spherical decreasing rearrangement $u^{\#}$ since by Hardy-Littlewood inequality and by Pólya-Szegö principle both terms of the inequality (6) decrease under spherical symmetrization. Our idea is then to fix not the rearrangement of u but the rearrangement of $|\nabla u|$. In this way both the L^2-norm and the L^q-norm of $|\nabla u|$ do not change and we have only to investigate what happens to the L^2-norm of u with weight $|x|^{-2}$. It can be seen that if $u \in H_0^1(\Omega)$ there exists a spherically symmetric function $\overline{u}$ defined on the ball $\Omega^{\#}$, such that $|\nabla \overline{u}|^{\#} = |\nabla u|^{\#}$ and $\int_\Omega \frac{u^2}{|x|^2} dx$ *increases* when we pass from u to $\overline{u}$. That will be enough for our purpose. Afterwards, we find the best value of the constant C in the inequality (6) in the case $q = 1$. Indeed in this case it is quite easy to see that, if u is a spherically symmetric function defined on a ball, the problem of finding the best value of the constant in the inequality (6) can be reduced to the study of the inequality (4).
It is clear that the same arguments apply when the L^q-norm of ∇u is replaced by a more general Lorentz norm $\||\nabla u|\|_{p,q}$ with $1 \leq q \leq p < 2$ (see section 3).

2 Best constant in Hardy- Sobolev inequalities with a remainder term in Lorentz spaces

In this section we prove inequalities (3) and (4). More precisely, we have the following

Theorem 1. *Let Ω be a bounded open set of $\mathbb{R}^N$ containing the origin, $N > 2$. The optimal value of the constant in the inequality (3) is given by*

$$C(|\Omega|) = \frac{\omega_N^{\frac{2}{N}}}{|\Omega|^{\frac{1}{N}}} V_0, \tag{7}$$

where ω_N is the measure of the N-dimensional unit ball and V_0 is the first zero of the function $V(r) = J_0(2\sqrt{r})$ (here J_0 denotes, as usual, the Bessel function of zero order).

Proof . As pointed out in the introduction it is enough to prove inequality (3) for spherically symmetric and decreasing functions defined on the ball $\Omega^{\#}$. Hence we are reduced to study the inequality

$$\int_{\Omega^{\#}} |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_{\Omega^{\#}} \frac{u^2}{|x|^2} dx \geq C \int_{\Omega^{\#}} \frac{u^2}{|x|} dx \qquad u \in H_0^1(\Omega^{\#}) \tag{8}$$

where u is a radial function. For simplicity we will study the inequality

$$\int_{B_R} |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_{B_R} \frac{u^2}{|x|^2} dx \geq \int_{B_R} \frac{u^2}{|x|} dx \qquad u \in H_0^1(B_R) \tag{9}$$

where B_R is the ball centered at the origin whose radius R must be determined and u is a radial function. This is not a restriction since (8) reduces to (9) using a suitable homothety. Indeed if $u = u(|x|)$ is a radial function for which (8) holds the function $z = u\left(\frac{|x|}{C}\right)$ satisfies (9) with $R = C\left(\frac{|\Omega|}{\omega_N}\right)^{1/N}$. We make the classical change of variable (5) which essentially allows us to read the Hardy-Sobolev inequality (9) as a Sobolev inequality in the plane. Indeed assuming $u \in C_0^1(B_R)$ and hence $v(0) = 0$, inequality (9) becomes

$$\int_0^R (v')^2 r\, dr \geq \int_0^R v^2\, dr \qquad v \in H^1(0,R),\ v(R) = 0, \tag{10}$$

Let us consider the functional

$$J(v) = \int_0^R (v')^2 r\, dr - \int_0^R v^2\, dr. \tag{11}$$

The Euler equation of this functional is

$$(rv')' + v = 0$$

a solution of this equation is the function

$$V(r) = J_0\left(2\sqrt{r}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{r^n}{(n!)^2}.$$

By standard arguments of Calculus of Variations it can be seen that the functions $v(r) = cV(r)$ with $c \in \mathbb{R}$ minimize the functional $J(v)$. Hence (10) holds for $R = V_0$ where $V_0 \simeq 1.4457\ldots$ is the first zero of the function $V(r) = J_0(2\sqrt{r})$.
Using again the change of variable (5) and coming back to the function u we obtain (9) for $R = V_0$. The restriction $u \in C_0^1(B_R)$ can be removed by density. A dimensional analysis on the constant shows that inequality (8) holds for $C = \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{1}{N}} V_0$. Hence the best value of the constant in the inequality (3) is given by (7). On the other hand it is clear that the optimal value of the constant is not attained since it would correspond to equality in (10) which happens for $v(r) = cV(r)$ and hence

$$u(x) = c|x|^{-\frac{N-2}{2}} V(|x|)$$

which is not in H^1. $\square$

An improvement of inequality (3) can be obtained by introducing a weighted norm of u with a logarithmic weight. The next proposition gives a very simple proof of this type of inequality. These inequalities are widely studied in [6], [16], [17].

Proposition 1. *Let Ω be a bounded open set of $\mathbb{R}^N$ containing the origin, $N > 2$, of measure $\frac{\omega_N}{e^N}$ and let p be a function defined in Ω whose spherical decreasing rearrangement is $p^{\#}(x) = \frac{1}{[|x| \log |x|]^2}$. Then the following inequality holds*

$$\int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx \geq \frac{1}{4} \int_\Omega p(x) u^2 dx. \tag{12}$$

Proof . The inequality (12) was first proved in [8] in the case $N = 1$ and more recently in [6] in any dimension. Pólya-Szegö principle and Hardy-Littlewood inequality allows us to reduce the study of (12) to radial function defined in the ball of radius $\frac{1}{e}$. In this case (12) becomes

$$\int_{B_{1/e}} |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_{B_{1/e}} \frac{u^2}{|x|^2} dx \geq \frac{1}{4} \int_{B_{1/e}} \frac{u^2}{[|x| \log |x|]^2} dx \tag{13}$$

We just observe that making the change of variable (5) with v such that $v(0) = v(1/e) = 0$ we get just the inequality proved in [8]. We give here an alternative and even simpler proof of (13). Making the change of variable

$$u(r) = v(r) r^{-\frac{N-2}{2}} \sqrt{-\log r}$$

with v such that $v(1/e) = 0$ and

$$\lim_{r \to 0} v(r) \log r = 0\,, \tag{14}$$

it results

$$\int_0^{1/e} (u')^2 r^{N-1} dr = \int_0^{1/e} (v')^2 |\log r| r dr + \frac{(N-2)^2}{4} \int_0^{1/e} \frac{v^2 |\log r|}{r} dr$$

$$+ \frac{1}{4} \int_0^{1/e} \frac{v^2}{r |\log r|} dr - (N-2) \int_0^{1/e} v v' |\log r| dr$$

$$- \int_0^{1/e} v v' dr + \frac{N-2}{2} \int_0^{1/e} \frac{v^2}{r} dr\,.$$

Using the boundary condition on v we have

$$\int_0^{1/e} v v' \log r dr = -\frac{1}{2} \int_0^{1/e} \frac{v^2}{r} dr\,.$$

Hence

$$\int_{B_{1/e}} |\nabla u|^2 dx \geq N\omega_N \frac{(N-2)^2}{4} \int_0^{1/e} \frac{v^2 |\log r|}{r} dr + \frac{1}{4} N\omega_N \int_0^{1/e} \frac{v^2}{r |\log r|} dr$$

that is (13).□

Much more interesting seems to us inequality (4) since it will be used in section 3 to find the best value of the constant in inequalities involving as a remainder term the $L(p,1)$ norm of the gradient for $0 < p < \infty$. We have

Theorem 2. *Let Ω be a bounded open set of $\mathbb{R}^N$ containing the origin, $N > 2$. Then for any $1 \le p < 2^*$, the optimal value of the constant in the inequality (4) is given by*

$$C(p, |\Omega|) = \frac{2\left(\frac{N}{p} - \frac{N}{2} + 1\right)^3}{N\,|\Omega|^{\frac{2\left(\frac{N}{p}-\frac{N}{2}+1\right)}{N}}} \cdot \omega_N^{\frac{2}{N}}. \tag{15}$$

Proof . As in the proof of theorem 1, using symmetrization we are reduced to prove inequality (4) for radial function u defined on $\Omega^{\#}$. Moreover if we replace u by the function $z = u\left(\frac{|x|}{K}\right)$ with $K^{-2(\frac{N}{p}-\frac{N}{2}+1)} = C$ we see that it is enough to study

$$\int_{B_R} |\nabla u|^2 dx - \frac{(N-2)^2}{4}\int_{B_R} \frac{u^2}{|x|^2}\,dx \ge \left(\int_{B_R} \frac{|u|}{|x|^{N-\frac{N}{p}}}\,dx\right)^2 \qquad u \in H_0^1(B_R) \tag{16}$$

where u is a radial function defined in a suitable ball B_R centered at the origin. Making the change of variable (5) and assuming $u \in C_0^1(B_R)$, we obtain the following inequality in the plane

$$\int_{C_R} |\nabla v|^2 dx \ge \frac{N\omega_N}{2\pi}\left(\int_{C_R} \frac{v}{|x|^{1-\frac{N}{p}+\frac{N}{2}}}dx\right)^2 \tag{17}$$

where C_R is the disk centered at the origin. Let us consider the functional

$$I(v) = \int_{C_R} |\nabla v|^2 dx - \frac{N\omega_N}{\pi}\int_{C_R} \frac{v}{|x|^{1-\frac{N}{p}+\frac{N}{2}}}dx = 2\pi\int_0^R (v')^2 r\,dr - 2N\omega_N\int_0^R \frac{v}{r^{\frac{N}{2}-\frac{N}{p}}}\,dr.$$

The function

$$V(r) = \frac{N\omega_N}{2\pi}\cdot\frac{1}{\left(\frac{N}{p}-\frac{N}{2}+1\right)^2}\left(R^{\frac{N}{p}-\frac{N}{2}+1} - r^{\frac{N}{p}-\frac{N}{2}+1}\right)$$

minimizes this functional. Moreover if

$$R^{2\left(\frac{N}{p}-\frac{N}{2}+1\right)} = \frac{2\left(\frac{N}{p}-\frac{N}{2}+1\right)^3}{N\omega_N}, \tag{18}$$

it satisfies

$$\int_{C_R} \frac{V}{|x|^{1-\frac{N}{p}+\frac{N}{2}}}dx = 1 \tag{19}$$

and

$$\int_{C_R} |\nabla V|^2 dx = \frac{N\omega_N}{2\pi}. \tag{20}$$

If v is any spherically symmetric function in $H_0^1(C_R)$ for which condition (19) holds, by (20) we have

$$\int_{C_R} |\nabla v|^2 dx - \frac{N\omega_N}{\pi} = I(v) \geq I(V) = -\frac{N\omega_N}{2\pi}$$

and then

$$\int_{C_R} |\nabla v|^2 dx \geq \frac{N\omega_N}{2\pi}.$$

Hence we find

$$\int_{C_R} |\nabla v|^2 \geq \frac{N\omega_N}{2\pi} \left(\int_{C_R} \frac{v}{|x|^{1-\frac{N}{p}+\frac{N}{2}}} \right)^2 \tag{21}$$

if we remove the assumption (19) on v. Coming back to the function u, using (21) we get (16) with R given in (18) . The assumption $u \in C_0^1(B_R)$ can be removed by density. A dimensional analysis on the constant shows that inequality (4) holds for $C(p, |\Omega|)$ given by (15).□

3 Best constant in Hardy-Sobolev inequalities with a remainder term depending on the gradient

In this section we focus our attention on the inequality (6). As pointed out in the introduction, our aim is to show a new approach that allows us to reduce the problem of finding the optimal value of the constant in (6) to a spherically symmetric situation. Our idea is to fix not the rearrangement of u, as we did in the proofs of the inequalities of the previous section, but the rearrangement of its gradient. In order to explain into details the result let us recall some preliminaries. Let f, g be two non negative measurable functions defined on two open bounded sets having the same measure V. We say that f is *dominated* by g, and we write $f \prec g$, if

$$\int_0^s f^*(\sigma) d\sigma \leq \int_0^s g^*(\sigma)\, d\sigma, \ \ \forall s \in [0, V]$$

and

$$\int_0^V f^*(\sigma) d\sigma = \int_0^V g^*(\sigma)\, d\sigma.$$

We say that f *is equimeasurable with* g, or that f *is a rearrangement of* g, if $f^* = g^*$. Let f_0 be a prescribed non negative decreasing and right-continuous function from $L^p(0, V)$, $p \geq 1$.

It is known that (see [5], [24] and also [15] for further details) if $f \in L^p(0,V)$ and $f \prec f_0$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(0,V)$ such that $f_n^* = f_0$ for all $n \in \mathbb{N}$ and f_n converges weakly to f in $L^p(0,V)$.

The following theorem shows that the value of the functional

$$J(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2} dx}{\||\nabla u|\|_q^2}$$

decreases if we replace Ω with the ball $\Omega^\#$, the function u by a suitable spherically symmetric and decreasing function $\overline{u} \in H_0^1(\Omega^\#)$ such that $|\nabla \overline{u}|$ is a rearrangement of $|\nabla u|$.

Theorem 3. *Let Ω be a bounded, open set of $\mathbb{R}^N$ containing the origin, $N > 2$. If $u \in H_0^1(\Omega)$ is a non negative function, then there exists a spherically symmetric decreasing function $\overline{u} \in H_0^1(\Omega^\#)$ such that $|\nabla \overline{u}|^* = |\nabla u|^*$ and*

$$J(u) \geq \frac{\displaystyle\int_{\Omega^\#} |\nabla \overline{u}|^2 \, dx - \frac{(N-2)^2}{4}\int_{\Omega^\#} \frac{\overline{u}^2}{|x|^2} dx}{\||\nabla \overline{u}|\|_q^2}. \tag{22}$$

Proof . In order to prove the theorem, it is enough to show that there exist a spherically symmetric decreasing function $\overline{u}$ as in the statement of the theorem, such that $\|u\|_{L(2^*,2)} \leq \|\overline{u}\|_{L(2^*,2)}$. A result of this type, when the norm involved is the L^q norm, with $1 \leq q \leq 2N/(N-2)$, is contained in [4]. Other related results are also contained in [13] and [27].
Set $f_0 = |\nabla u|^*$. By a result due to [20] (see also [4]) we get that

$$u^*(s) \leq \frac{1}{N\omega_N^{\frac{1}{N}}} \int_s^{|\Omega|} \frac{F(t)}{t^{1-\frac{1}{N}}} dt, \quad \forall s \in [0, |\Omega|], \tag{23}$$

for some non negative function $F \in L^2(0,|\Omega|)$, $F \prec f_0$. The right-hand side of (23) is the unique spherically symmetric decreasing solution to the problem

$$\begin{cases} |\nabla g| = F(\omega_N |x|^N) \ \text{ in } \Omega^\# \\ \\ g = 0 \ \text{ on } \partial\Omega^\#. \end{cases}$$

i.e.

$$g(|x|) = \int_{|x|}^{\left(\frac{|\Omega|}{\omega_N}\right)^{\frac{1}{N}}} F\left(\omega_N s^N\right) ds.$$

The above result allows us to say that

$$\|u\|_{L(2^*,2)(\Omega)} \leq \|g\|_{L(2^*,2)(\Omega^\#)}. \tag{24}$$

At this point the proof is based on a duality argument, an approach that is different from the one shown in [4]. For each $F \in L^2(0,|\Omega|)$, $F \prec f_0$, we define

$$I(F) = \|g\|_{L(2^*,2)\left(\Omega^{\#}\right)} = \left[\int_0^{|\Omega|}\left(\frac{1}{N\omega_N^{1/N}}\int_s^{|\Omega|}\frac{F(t)}{t^{1-1/N}}dt\right)^2 s^{-2/n}ds\right]^{1/2}.$$

On the other hand, since the dual of $L(2^*,2)\left(\Omega^{\#}\right)$ is the Lorentz space $L\left(\frac{2N}{N+2},2\right)\left(\Omega^{\#}\right)$, by definition of the norm in the dual space, for a fixed $F \in L^2(0,|\Omega|)$, $F \prec f_0$, we find

$$I(F) = \max_{\substack{\phi\in L\left(\frac{2N}{N+2},2\right)\\ \|\phi\|_{\frac{2N}{N+2},2}=1}} \frac{1}{N\omega_N^{1/N}}\int_0^{|\Omega|}\phi(s)\left(\int_s^{|\Omega|}\frac{F(t)}{t^{1-1/N}}dt\right)ds. \tag{25}$$

Let ϕ be a function for which the maximum in (25) is attained. Integrating by parts we obtain

$$\int_0^{|\Omega|}\phi(s)\left(\int_s^{|\Omega|}\frac{F(t)}{t^{1-1/N}}dt\right)ds = \int_0^{|\Omega|}\frac{F(s)}{s^{1-1/N}}\left[\int_0^s\phi(t)\,dt\right]ds.$$

Hence

$$I(F) = \int_0^{|\Omega|}\frac{F(s)}{N\omega_N^{1/N}s^{1-1/N}}\left[\int_0^s\phi(t)\,dt\right]ds.$$

Let

$$\psi(s) = \frac{1}{N\omega_N^{1/N}s^{1-1/N}}\int_0^s\phi(t)\,dt,$$

then $\psi \in L^2(0,|\Omega|)$. Indeed

$$\begin{aligned}\left(\int_0^{|\Omega|}[\psi(s)]^2\,ds\right)^{1/2} &= \frac{1}{N\omega_N^{1/N}}\left(\int_0^{|\Omega|}s^{-2+\frac{2}{N}}\left(\int_0^s\phi(t)\,dt\right)^2ds\right)^{1/2}\\ &= \frac{1}{N\omega_N^{1/N}}\left(\int_0^{|\Omega|}s^{\frac{2}{N}}\left(\frac{1}{s}\int_0^s\phi(t)\,dt\right)^2ds\right)^{1/2} \le \frac{1}{N\omega_N^{1/N}}\|\phi\|_{\frac{2N}{N+2},2}.\end{aligned}$$

Since $F \in L^2(0,|\Omega|)$ and $F \prec f_0$, we can find a sequence (see [5], [24])

$$\{f_n\}_{n\in\mathbb{N}} \subseteq L^2(0,|\Omega|),\ f_n^* = f_0$$

such that $f_n \underset{n}{\rightharpoonup} F$ weak in $L^2(0,|\Omega|)$, then

$$I(F) = \lim_{n\to\infty}\int_0^{|\Omega|}f_n(s)\,\psi(s)\,ds.$$

By Hardy-Littlewood inequality

$$\int_0^{|\Omega|} f_n(s)\,\psi(s)\,ds \leq \int_0^{|\Omega|} f_0(s)\,\psi^*(s)\,ds,$$

moreover (as it is shown in [12]), it is possible to construct a rearrangement $\overline{f}_\psi \in L^2(0,|\Omega|)$ of f_0 such that

$$\int_0^{|\Omega|} f_0(s)\,\psi^*(s)\,ds = \int_0^{|\Omega|} \overline{f}_\psi(s)\,\psi(s)\,ds.$$

The function $\overline{f}_\psi$ is obtained by taking a sort of mean value of f_0, it essentially can be expressed in terms of the mean value operator introduced in [24] and is connected with the notion of pseudo-rerrangement or relative rearrangement (see also [4]).

Therefore for any $F \in L^2(0,|\Omega|)$ such that $F \prec f_0$ we have

$$I(F) \leq \int_0^{|\Omega|} \overline{f}_\psi(s)\,\psi(s)\,ds. \tag{26}$$

Recalling the definition of ψ, an integration by parts allows us to get that

$$\begin{aligned}\int_0^{|\Omega|}\overline{f}_\psi(s)\,\psi(s)\,ds &= \int_0^{|\Omega|} -\frac{d}{ds}\left(\int_s^{|\Omega|}\frac{\overline{f}_\psi(t)}{N\omega_N^{\frac{1}{N}}t^{1-\frac{1}{N}}}dt\right)\left(\int_0^s \phi(t)\,dt\right)ds \\ &= \frac{1}{N\omega_N^{\frac{1}{N}}}\int_0^{|\Omega|}\phi(s)\left(\int_s^{|\Omega|}\frac{\overline{f}_\psi(t)}{t^{1-\frac{1}{N}}}dt\right)ds = I\left(\overline{f}_\psi\right)\end{aligned} \tag{27}$$

Hence setting

$$\overline{u}(|x|) = \int_{|x|}^{\left(\frac{|\Omega|}{\omega_N}\right)^{\frac{1}{N}}} \overline{f}_\psi\left(\omega_N s^N\right)ds,$$

by (24), (26), (27) we have found a spherically symmetric decreasing function defined on $\Omega^\#$ such that $|\nabla\overline{u}|^* = f_0$ and for which

$$\|u\|_{L(2^*,2)(\Omega)} \leq \|\overline{u}\|_{L(2^*,2)\left(\Omega^\#\right)},$$

that is inequality (22) holds.□

Now if we come back to the problem concerning the calculation of the best constant C in (6), i.e. the infimum

$$C = \inf_{\substack{u\in H_0^1(\Omega)\\ u\neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2 - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}dx}{\||\nabla u|\|_q^2}, \tag{28}$$

theorem 3 allows us to restrict our attention to the class of spherically symmetric decreasing functions $u \in H_0^1(\Omega^\#)$.

Indeed, if we reduce the study to spherically symmetric decreasing functions $u \in H_0^1(B_R)$ with $R = (|\Omega| / \omega_N)^{1/N}$, integrating by parts we have

$$\int_{B_R} |\nabla u| \, dx = -N\omega_N \int_0^R u'(r) \, r^{N-1} dr = N\omega_N \left[\lim_{r \to 0} u(r) \, r^{N-1} + (N-1) \int_0^R u(r) \, r^{N-2} dr \right].$$

Since H_0^1 is imbedded $L(2^*, \infty)$ we find that, for a suitable constant k

$$u(r) \leq k r^{-\frac{N-2}{2}},$$

from which it follows

$$\lim_{r \to 0} u(r) \, r^{N-1} = 0.$$

Hence

$$\int_{B_R} |\nabla u| \, dx = (N-1) \int_{B_R} \frac{u}{|x|} dx = (N-1) \, \omega_N^{\frac{1}{N}} \, \|u\|_{\frac{N}{N-1},1} \, .$$

Applying theorem 2 with $p = N/(N-1)$ and hence $C(p, |\Omega|) = N^2 \omega_N^{2/N} / (4|\Omega|)$, we deduce

$$\begin{aligned} C &= \frac{1}{\omega_N^{2/N} \, (N-1)^2} \inf_{\substack{u \in H_0^1(\Omega^\#) \\ u = u^\# \\ u \not\equiv 0}} \frac{\displaystyle\int_{\Omega^\#} |\nabla u|^2 \, dx - \frac{(N-2)}{4} \int_{\Omega^\#} \frac{u^2}{|x|^2} dx}{\|u\|^2_{\frac{N}{N-1},1}} \\ &= \frac{1}{4\,|\Omega|} \left(\frac{N}{N-1} \right)^2 . \end{aligned}$$

In conclusion we have

Theorem 4. *Let Ω be a bounded open subset of $\mathbb{R}^N$ containing the origin, $N > 2$. If $q = 1$, the optimal value of the constant in inequality (6) is given by*

$$C(|\Omega|) = \frac{1}{4\omega_N \, |\Omega|} \left(\frac{N}{N-1} \right)^2 .$$

It is clear that it is always possible to reduce the calculation of the best constant C to a spherically symmetric situation in inequalities of the type (5) that involve the Lorents norms $\||\nabla u|\|_{p,q}$ with $1 \leq q < p < 2$ or $\||\nabla u|\|_{p,1}$ with $0 < p < 1$. In

particular, the method used in the proof of theorem 4 allows us to treat inequalities of the type (6) when the right hand side is

$$\int_{\Omega^\#} |\nabla u|^\# |x|^\alpha \, dx, \text{ with } \alpha > 0,$$

that is essentially a norm of $|\nabla u|$ in a Lorentz space $L(p,1)$ with $0 < p = \frac{N}{N+\alpha} < 1$. Indeed, also in this case, by an integration by parts, the problem is reduced to the study of an inequality of the type (4). In fact, using the Hardy-Littlewood inequality, since the function $|x|^\alpha$ is increasing we find

$$\int_{\Omega^\#} |\nabla u|^\# |x|^\alpha \, dx \le \int_{\Omega^\#} |\nabla u| \, |x|^\alpha \, dx = (N+\alpha-1) \int_{\Omega^\#} u \, |x|^{\alpha-1} \, dx \tag{29}$$

for all $u \in H_0^1\left(\Omega^\#\right)$ such that $u = u^\#$. More precisely, the following theorem can be proved:

Theorem 5. *Let Ω be a bounded open subset of $\mathbb{R}^N$ containing the origin, $N > 2$. If $q = 1$ and $\max\left\{\frac{2N}{3N-2}, \frac{N}{N+1}\right\} < p < 1$, the optimal value of the constant in the inequality*

$$\int_\Omega |\nabla u|^2 \, dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx \ge C \, \| |\nabla u| \|_{p,1}^2 , \quad \forall u \in H_0^1(\Omega), \tag{30}$$

is given by

$$C(p, |\Omega|) = \frac{\left(\frac{2-p}{p}\right)^3}{4 \, |\Omega|^{2/p-1}} \left(\frac{Np}{N-p}\right)^2 .$$

References

[1] ADIMURTHI, N. CHAUDHURI, M. RAMASWAMY: *An improved Hardy-Sobolev inequality and its application*, Proc. Amer. Math. Soc. **130** (2002), 485-505.

[2] ADIMURTHI, SANDEEP: *Existence and nonexistence of eigenvalue of the perturbed Hardy-Sobolev operator*, Proc. Royal Soc. Ed. Sec. A.132 **5** (2002), 1021-1043.

[3] A. ALVINO: *Sulla diseguaglianza di Sobolev in spazi di Lorentz*, Boll. U.M.I., (5) **14**-A (1977), 148-156.

[4] A. ALVINO, P.L. LIONS, G. TROMBETTI: *On optimization problems with prescribed rearrangements*, Nonlinear Anal. (2) **13** (1989), 185–220.

[5] A. ALVINO, G. TROMBETTI: *Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri*, Ricerche Mat. **27** (1978), 413–428.

[6] G. BARBATIS, S. FILIPPAS, A. TERTIKAS: *A unified approach to improved L^p Hardy inequalities with best constants*, Trans. Amer. Math. Soc., (6) **356** (2004), 2169–2196

[7] C. BENNET, R. SHARPLEY: Interpolation of Operators, Pure and Appl. Math. Vol. 129, Academic Press, 1988.

[8] H. BREZIS, M. MARCUS: *Hardy's inequalities revisited*, Ann. Sc. Norm. Super. Pisa Cl. Sci.(4) **25** (1997), no. 1-2, 217-237 (1998).

[9] H. BREZIS, M. MARCUS, I. SHAFRIR: *Extremal functions for Hardy's inequality with weight*, J. Funct. Anal **171** (2000), 177-191.

[10] H. BREZIS, J.L. VAZQUEZ: *Blow - up solutions of some nonlinear elliptic problems,* Revista Matemática de la Universidad Complutense de Madrid, (10) **2** (1997), 443-469.

[11] N. CHAUDURI, M. RAMASWAMY: *Existence of positive solutions of some semilinear elliptic equations with singular coefficients*, Proc. Roy. Soc. Edinburgh Sect. A 131 **6** (2001), 1275–1295.

[12] K. M. CHONG, N. M. RICE: *Equimeasurable rearrangements of functions*, Queen's Papers in Pure and Applied Mathematics, No. 28. Queen's University, Kingston, Ont., 1971.

[13] A. CIANCHI: *On the $L(p,q)-$norm of functions having equidistributed gradients*, Nonlinear Anal. (12) **26** (1996), 2007–2021.

[14] A. FERONE - R. VOLPICELLI: *Some relations between pseudo-rearrangement and relative rearrangement*, Nonlinear Anal., **41** (2000) n.7-8, 855-869.

[15] A. FERONE, R. VOLPICELLI: *Polar factorization and pseudo-rearrangements: applications to Pólya-Szegö type inequalities*, Nonlinear Anal. **53** (2003), no.7-8, 929–949.

[16] S. FILIPPAS, V. G. MAZ'YA, A. TERTIKAS: *Sharp Hardy-Sobolev inequalities*, C. R. Math. Acad. Sci. Paris (7) **339** (2004), 483–486.

[17] S. FILIPPAS, A. TERTIKAS: *Optimizing improved Hardy inequalities*, J. Funct. Anal. (1) **192** (2002), 186–233.

[18] F. GAZZOLA, H. C. GRUNAU, E. MITIDIERI: *Hardy inequalities with optimal constants and remainder terms*, Trans. Amer. Math. Soc., (6) **356** (2004), 2149–2168.

[19] N. GHOUSSOUB, A. MORADIFAM: *On the best possible remaining term in the Hardy inequality*, preprint arXiv: math/0703506v1 [math.AP], 16 Mar. 2007.

[20] E. GIARRUSSO, D. NUNZIANTE: *Symmetrization in a class of first-order Hamilton-Jacobi equations*, Nonlinear Anal., (4) **8** (1984), 289–299.

[21] E. GIARRUSSO, D. NUNZIANTE: *Comparison theorems for a class of first order Hamilton-Jacobi equations*, Ann. Fac. Sci. Toulouse Math. (5), (1) **7** (1985), 57–73.

[22] G. H. HARDY: *Notes on some points in the integral calculus*, Messenger Math. **48** (1919), 107-112.

[23] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA: *Inequalities*, Cambridge University Press, 1934.

[24] J. MOSSINO, R. TEMAM: *Directional derivative of the increasing rearrangement mapping, and application to a queer differential equation in plasma physics*, Duke Math. J., **48** (1981), 475-495.

[25] G. PÒLYA, G. SZEG"O: *Isoperimetric inequalities in mathematical physics*, Ann. of Math. Studies, 27, Princeton University Press, Princeton, 1951.

[26] G. TALENTI: *Best Constant in Sobolev Inequality*, Ann. Mat. Pura Appl., **110** (1976), 353-372.

[27] G. TALENTI: *On functions whose gradients have a prescribed rearrangement*, Inequalities and applications, 559–571, World Sci. Ser. Appl. Anal., 3, World Sci. Publ., River Edge, NJ, 1994.

[28] J. L. VAZQUEZ, E. ZUAZUA: *The Hardy Inequality and the Asymptotic Behaviour of the Heat Equation with an Inverse-Square Potential,* Journal of Functional Analysis, **173** (2000), 103-153.